

Holographic bounds and finite inflation

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We compare two holographic arguments that impose especially strong bounds on the amount of inflation. One comes from the de Sitter Equilibrium cosmology and the other from the work of Banks and Fischler. We find that simple versions of these two approaches yield the same bound on the number of e-foldings. A careful examination reveals that while these pictures are similar in spirit, they are not necessarily identical prescriptions. We apply the two pictures to specific cosmologies which expose potentially important differences and which also demonstrate ways these seemingly simple proposals can be tricky to implement in practice.

I. INTRODUCTION

There have been a number of attempts to apply notions of holography towards constraints on inflation. [1–3]. The motivation is simple: holography implies an encoding of bulk information in correlations on a boundary, while inflation promotes quantum fluctuations to all scales and (while it lasts) seems to provide a fertile source for new information. Where these notions come in conflict we can look to place limits on the amount of inflation allowed; but the strength and form of the limits will depend strongly on the particular holographic approach adopted.

Holographic limits on inflation are of particular interest in the context of “eternal inflation” [4–6], which uses simple extrapolations within effective field theory (EFT) to suggest that at some high level inflation continues forever, giving birth to unbounded numbers of “pocket universes”. Holographic limits may give hints about how a deeper theory would lead to a breakdown of the EFT, and perhaps dramatically alter the eternal inflation picture (and perhaps resolve the notorious measure problems associated with eternal inflation)[7, 8].

One example of an inflation model limited by holographic arguments is the de Sitter Equilibrium (dSE) picture [1, 8, 9]. In that picture the universe is fundamentally finite, with a maximum entropy associated with the asymptotic de Sitter horizon at late times. Because of the finite Hilbert space, the standard EFT description of inflation will fail if asked to model a length of inflation producing enough volume to exceed the universe’s maximum information content. In [1] this bound manifests as a sharp prediction for spatial curvature of our universe, as a function of initial bubble curvature. In this paper we will show the bounds achieved in dSE are identical to those found by Banks and Fischler (BF) [3] despite a treatment that incorporates holography differently.

However, we show that specific assumptions chosen by BF in addition to their maximum-entropy method of deriving a bound on inflation are needed together to enforce the same geometric principles used in the dSE curvature prediction. As demonstrated in [10], adjusting those assumptions to ones more representative of our own universe can modify the prediction for a maximum number of e-foldings of inflation. Although not immediately apparent, some of those modifications would result in different physical pictures and indeed could produce different bounds from the dSE case. Each picture has originally been presented in fairly simple terms, and our work exposes ways the simple definitions appear to be insufficient to allow for a full implementation in all cases. This is how the additional assumptions can become especially important. It appears that a geometric interpretation of the holographic principle along the lines of [1] is useful to clarify a number of these issues.

Our paper is organized as follows. In Sec. II we quickly present versions of each picture (BF¹ and dSE) with many simplifying cosmological assumptions and approximations. As we show, these approximations cause the two pictures to converge not only on each other, but also on cosmologies with past histories similar to ours (thus suggesting that these ideas are quite relevant to our universe). In Secs. III and IV we study this apparent correspondence. In

¹ We will repeatedly refer to the result found in the first half of the paper by Banks and Fischler [3] as the BF picture. The project of our paper is a comparison between the holographic principle underlying this particular result and the one within the dSE picture, and to that purpose we will add specifications, and reflect upon assumptions and motivations for the “BF picture”. These reflections and modifications are entirely those of this paper’s authors and we do not mean to attribute them to Banks and Fischler.

Sec. V we begin to tease apart the two pictures by examining the assumptions required for each. In the first step we simply ask what additional clarifications are needed in each picture for applications to more realistic and specific cosmologies. It is generally possible to ensure that the BF and dSE pictures return the same results, but the process of picking the “correct” choice of assumptions to match the two begins to look ad hoc. Finally we examine the two principles with cosmologies that are parametrically connected to ours but substantially altered. In this manner we can better expose conceptual differences between the two approaches when pushed away from their convergence near our own relatively simple cosmological history.

II. HOLOGRAPHY IN DSE AND BF

A. General discussion

In both the dSE and BF pictures, the future of our universe is asymptotically de Sitter, with a fundamental cosmological constant Λ . In both of these pictures the entropy $3\pi/\Lambda$ associated with the de Sitter horizon represents the finite amount of information associated with the entire universe. In such a picture it is expected that physics can describe semiclassical spacetimes with a maximum of one horizon volume (although different observers can observe different realizations of such a volume by swapping out information encoded non-locally at the de Sitter horizon with the interior).

Though the horizon entropy is important in the setup of dSE cosmology, Banks and Fischler do not use it in their calculation of the bound on the number of e-foldings. Instead, the universe is modeled as a fluid-filled cavity the size of the apparent horizon, and the entropy within that cavity is used to obtain the bound.

It is important to distinguish between these holographic principles and other variants that physically differ and result in different (or no) bounds on the amount of inflation. For example, the covariant Bousso entropy bound [11] is formulated on the past light cone of an observer. Placing such bounds on the past light cone does not restrict the number of e-foldings of inflation [12]. In a similar manner Kaloper *et al.* [13] interpret the BF bound as only placing limitations on the number of e-foldings of inflation that will ever be observable. In contrast, in the dSE picture the entire universe is eventually observable so there is no distinction to be made be-

tween observable and total e-foldings.

Numerous authors have proposed bounds on inflation under a variety of assumptions. For example, Arkani-Hamed *et al.* found a much less stringent bound of $N_{total} < S$ (where S is the entropy of the final asymptotic de Sitter space) by demanding non-eternal inflation [2]. Albrecht *et al.* also put forward another holographic inflation bound by using the slowly changing apparent horizon to estimate the entropy of metric fluctuations expelled during the slow roll period of inflation [14]. Bousso’s D-bound originates from the requirement that entropy not decrease during the transition to empty de Sitter space and works by positing that the entropy gained from the increased horizon area must exceed that of the matter entropy that was lost [15]. These bounds each emerge from fundamentally different holographic principles and we feel each is interesting in its own right. Here we restrict our attention only to the bounds from the dSE and BF pictures because they seem to admit direct comparison.

Finally, for a given holographic principle it is important to distinguish between an absolute bound on the length of inflation for any universe (e.g. allowing for the most extreme variations in reheating, matter fraction, etc. consistent with some set of cosmological assumptions), and a bound on inflation for a universe consistent with the one we observe. The first bound is of more interest for exploring a multiverse of cosmologies, either to understand the allowable regions within a theory or for making predictions within a multiverse. Constructing the second type of bound is more relevant for direct connections to observations. For now we will restrict ourselves to simplifying assumptions relevant for our universe, but later (in Sec. V) we will expand our focus to a broader range of cosmologies.

B. dSE bound basics

Under the de Sitter Equilibrium picture’s assumption of a finite universe whose size is set by a fundamental cosmological constant [8], an observer near the universe’s final approach towards de Sitter space should be able to see essentially all that there is in the universe. Requiring the past horizon of such a “maximal observer” to contain all scales produced by inflation puts a bound on the maximal length of inflation, which could otherwise generate structure that grew to physical scales arbitrarily larger than the universe’s size by the time of the maximal observer.

C. Banks Fischler bound basics

Banks and Fischler [3] follow the entropy to examine how an ultimate size limit for the universe restricts the length of inflation. They note a number of results restricting the maximum entropy within a sphere for a non-collapsing fluid with a given equation of state, and then demand that the ultimate entropy of the universe be no larger than that limit calculated for a sphere of the universe’s ultimate size, for the appropriate fluid. As an initial patch inflates its volume increases, so the limit on the length of inflation arises by requiring that upon reheating the total entropy of the entire inflated region does not exceed this calculated fundamental limit.

D. The connection between the two pictures

The BF and dSE pictures use holography-inspired principles that are very similar and amount to restricting the ultimate radius of the universe to approximately the de Sitter radius. In both pictures, by the onset of de Sitter domination we expect an observer to be able in principle to observe everything produced during inflation, rather than allowing some matter to remain forever out of reach. From these two facts it appears as though the two approaches are guaranteed to deliver essentially the same bounds on inflation, and to a certain degree this is true. However, as we will see in Sec. V, the pictures are not necessarily physically identical, and can only be made so with the buttressing of enough simplifying assumptions to force the two pictures to describe identical scenarios. The logical differences between the two remain of interest and investigating what is required to bring the two into alignment helps to clarify both pictures.

E. Counting e-foldings in dSE: A geometrical picture

Figure 1 shows the evolution of the Hubble radius $R_H \equiv cH^{-1}$. Also shown is h_P , the past horizon of an event late in the cosmological constant dominated regime. Specifically, if the event is the observation of a photon which travels freely before the observation, h_P is the distance between the photon and the observer prior to observation. Owing to the formation of an event horizon as we approach a de Sitter background, the past horizons of events at any time in the de Sitter regime are much the same. Any of

these observers can see essentially as much volume as any observer ever will. Even today ($a = a_0$) we are not too far off from being such “maximal observers”, since Λ is already quite dominant. The cosmological parameters for the curves shown match our universe with reasonably fast reheating assumed.

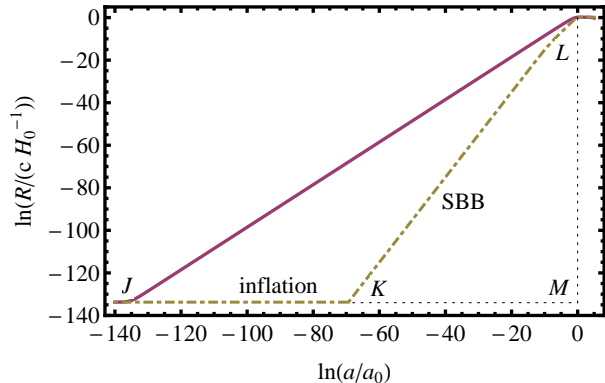


FIG. 1: The evolution of various length scales with scale factor a plotted during inflation and the subsequent standard big bang expansion (SBB) up to the dS scale for a simple model of our universe. Plotted are the Hubble length (dot-dashed) and the past horizon h_P of an event late in the de Sitter era (solid). The radiation-matter transition occurs at $\ln(a/a_0) = -8.1$ and shows up as a very slight kink in the dot-dashed curve. Markers J, K, L and M aid the discussion in the text.

The formulation of dSE cosmology [1] requires the entire universe to be contained within the past horizon of such a maximal observer. If our classical description of inflation begins earlier than shown in Fig. 1, there would be scales which never re-entered the horizon of this maximal observer, and which would represent physical scales larger than the finite size ($\sim \Lambda^{-1/2}$) assumed for the universe. This requirement places a limit on the amount of inflation allowed in dSE.

The form of the dSE constraint leads to a particularly simple connection with the cosmic curvature. Through most of the universe’s history, the past horizon scale evolves $\sim a$. Because the spatial curvature radius also evolves $\sim a$, one can make a sharp prediction for today’s measured spatial curvature density Ω_k in terms of the spatial curvature of the bubble that began inflation [1]. Due to the geometric nature of this picture the details of reheating and the subsequent evolution of the universe can modify the length of inflation, but they do not change the prediction for the curvature. Thus curvature is a more robust reflection of the dSE bound

than number of e-foldings. Nonetheless, one can apply this geometric framework to derive bounds on the length of inflation as well.

The geometric notions in dSE allow us to calculate the bound on the length of inflation in the dSE picture because the curvature prediction amounts to considering the geometry of a plot such as Fig. 1. For simplicity we'll approximate the evolution of the physical photon distance by two line segments (replacing the smooth transition region near $a = a_0$ with a sharp corner). For the segment representing past evolution, the distance evolution is proportional to a , so its slope is unity in Fig. 1. We will also assume that during inflation the Hubble parameter H_I is approximately constant and the subsequent reheating is rapid. After reheating we treat the universe as a perfect fluid with a single equation of state $p = w\rho$ right up to the beginning of the dS stage (which sets in with $H = H_\Lambda$ at scale factor a_Λ). Thus we are also representing the R_H curve in Fig. 1 with straight line segments meeting at sharp corners. Since most of the logarithmic range of a is in the radiation era (vs matter era), and the slope of the R_H curve during the matter era is not that different from the radiation case, one can achieve a good approximation to Fig. 1 by taking $w = 1/3$ in our linear approximation.

With these simplifications and the dSE assumption fixing the beginning of inflation on the past horizon line, the increase in the Hubble radius H^{-1} from inflation to dS must equal the increase in the scale factor over the same time: $\ln \frac{H_\Lambda^{-1}}{H_I^{-1}} = \ln \frac{a_\Lambda}{a_I}$. On Fig. 1 this corresponds to setting the lengths $JM = LM$.

To find the number of e-foldings for inflation (JK), we simply start with the e-fold increase in the Hubble length (JM) and subtract off the e-folds of a “eaten up” during the standard big bang expansion (SBB) of the fluid to the dS scale (KM). Since $H^{-1} \sim \rho^{-1/2}$ and $\rho \sim a^{-3(1+w)}$, $H^{-1} \sim a^{3(1+w)/2}$, which gives us the slope during SBB. Thus the increase of a during SBB is given by $\frac{2}{3(1+w)} \ln \frac{H_I}{H_\Lambda}$, and the number of e-foldings N_e of inflation is

$$N_e = JM - KM \quad (1)$$

$$N_e = \ln \frac{H_I}{H_\Lambda} \left(1 - \frac{2}{3(1+w)}\right) \quad (2)$$

$$= f \ln \frac{H_I}{H_\Lambda}, \quad f \equiv \frac{1}{3} \left(3 - \frac{2}{(1+w)}\right) \quad (3)$$

In the case of radiation ($w = 1/3$), $f = 1/2$ and the universe's history is evenly split: the magnitude of the universe's expansion during inflation equals that from reheating through the start of Λ domina-

tion. For GUT-scale $H_I \approx 10^{40} \text{ s}^{-1}$ and the observed $H_\Lambda \approx 10^{-18} \text{ s}^{-1}$, that gives $N_e = \frac{1}{2} \ln \frac{H_I}{H_\Lambda} \approx 67$ e-foldings. Decreasing H_I will increase R_H during inflation, raising the horizontal line segment in Fig. 1, and thus decreasing the bound on N_e , assuming the other elements of the calculation are held fixed.

III. ENTROPY BOUNDS FOR FLUIDS SYSTEMS

A. Overview

Banks and Fischler's holographic approach involves counting the entropy in a cavity the size of the apparent horizon. With this method of entropy counting, putting a bound on inflation involves putting a bound on the entropy within this cavity. Thus we will find it useful to discuss bounds on the amount of entropy that can be contained within a cavity of size R without it collapsing due to gravity. Banks and Fischler examine a number of approaches to bound the entropy of a fluid system. In each case the scaling obtained for fluids with equation of state $p = w\rho$ is of the form

$$S \leq \beta \left(\frac{R}{l_p}\right)^{3 - \frac{2}{1+w}}, \quad (4)$$

where l_p is the Planck length and β is a constant determined by the thermodynamic relation for the entropy density of the fluid, discussed in more detail in Sec. V C. For radiation, $\beta = O(1)$ in these units one Planck volume should contain at most roughly one unit of entropy. The common form (Eqn. 4) for these results formed the basis of the derivation for the BF inflation bound. Banks and Fischler derived their bound in a simple cosmology and we next reproduce their argument in the remainder of this section.

B. Critical (flat) FRW universes without dark energy

We write the Friedmann equation as

$$H^2 = \frac{8\pi G}{3} \rho_{tot} - \frac{k}{a^2}. \quad (5)$$

Using the thermodynamic relation for the entropy density

$$\sigma = \beta \rho^{\frac{1}{1+w}} \quad (6)$$

of a fluid with equation of state $p = w\rho$, we can calculate the total entropy contained within a Hubble volume H^{-3} for a flat ($k = 0$) universe:

$$S = H^{-3}\sigma = \beta H^{-3}\rho^{\frac{1}{1+w}} = \beta H^{-3+\frac{2}{1+w}} \quad (7)$$

The factor β is at most $O(1)$ for a single species (for σ and ρ expressed in Planck units) but can be significantly smaller, as discussed in Sec. V C.

C. Universes with $\Lambda > 0$

Similar results exist for certain cases in universes with a cosmological constant [3]. Fischler et al. [16] consider how much entropy can be stuffed into a region without it collapsing by relating the energy density to the entropy density using the thermodynamic relation (6), and then solving the Friedmann equations. They point out that for static solutions in matter-dominated ($w = 0$) universes, there is a metric sign change at precisely the same entropy bound given by Eqn. (4). Fischler et al. show that violating this bound in the collapsing phase of a universe with positive Λ causes a big crunch [16].

In both dSE and the BF approach the finite size of the universe places an upper limit on the radius of a fluid-filled sphere. This maximum radius would then imply a maximum theoretical entropy for such a fluid-filled universe. The repeated appearance of the relation (4) encouraged BF [3] to ask what limitations on inflation could result if one demands the global entropy produced during inflation to remain less than this entropy bound evaluated at the maximal radius $\sqrt{3/\Lambda}$. Our paper seeks to compare the resulting bound on inflation with related results within dSE.

D. BF e-fold counting

Banks and Fischler [3] arrived at a formula for e-foldings identical to Eqn. 1 in the first half of their paper.² BF adopt a holographically inspired view that functionally bounds space within a cavity of radius $\Lambda^{-\frac{1}{2}}$. As discussed in Sec. III B, filling a cavity of this size with a fluid of equation of state $p = \rho w$

results in a maximum entropy $S \lesssim R^{3-\frac{2}{1+w}}$. Their approach is to assume that the inflaton reheats into a fluid with state parameter w at a density $\rho_i \sim H_I^2$, with an entropy density $\sigma_i = \beta\rho_i^{\frac{1}{1+w}}$. At the reheating time there are e^{3N_e} Hubble patches. The goal is to ensure that if we sum all the entropy in these patches, we do not exceed the limit

$$S_{max} \sim H_\Lambda^{-(3-\frac{2}{1+w})} \quad (8)$$

for the asymptotic apparent horizon $\sim H_\Lambda^{-1}$.

The entropy S_i in a single Hubble volume after reheating is

$$S_i = H_I^{-3}\sigma_i \quad (9)$$

$$= H_I^{-3}\beta\rho_i^{\frac{1}{1+w}} \quad (10)$$

$$\sim H_I^{-(3-\frac{2}{1+w})} \quad (11)$$

where H_I is the value of the Hubble constant at the end of inflation and (as in [3]) we have suppressed pre-factors such as β . Thus the entire volume of e^{3N_e} Hubble patches must obey

$$e^{3N_e}S_i \leq S_{max} \quad (12)$$

$$e^{3N_e}H_I^{-(3-\frac{2}{1+w})} \leq H_\Lambda^{-(3-\frac{2}{1+w})} \quad (13)$$

$$(14)$$

giving

$$N_e \leq \frac{1}{3}\left(3 - \frac{2}{1+w}\right) \ln \frac{H_I}{H_\Lambda} \quad (15)$$

which is identical to the dSE case (Eqn. 1).

IV. HOW THE BF AND DSE BOUNDS WIND UP THE SAME

We would like to relate the BF result as closely as possible to the dSE result. We can phrase the dSE bound most simply as the requirement that the increase in physical volume a^3 from the beginning of inflation to the beginning of the dS era equals the increase in Hubble volume over the same period. At first glance the BF bound does not depend on such geometric notions; it merely demands the global entropy produced at reheating be no more than the maximum allowed for a fluid that can fill a cavity the size of the de Sitter horizon without collapse.

In order to connect the BF picture and the geometric ideas from dSE one can follow a comoving region of space through the evolution of the universe.

² With the addition of a ‘‘holographic gauge’’ condition, they later produce a different estimate. As dSE cosmology does not incorporate an analogous condition, we will restrict our comparison to the first estimate.

We focus on the region C bounded by the apparent horizon when inflation starts. (The dashed line in Fig. 2 shows the size of region C.) During inflation, C expands exponentially, but the size of the apparent horizon A stays the same. After reheating the resulting radiation dominated universe C expands too, but the region that is contained within the apparent horizon A expands faster. This is also the case during matter domination. When we reach cosmological constant domination, both A and C have become very large and we put a bound on inflation by requiring A to be contained within C at all times.³ Since C describes a comoving region the entropy in this region is conserved assuming adiabatic evolution.

Because the underlying restriction on the size of the universe is the same in both approaches, it may seem that the bounds on inflation are destined to be identical. But while this restriction is explicit in the derivation for the dSE case, it does not appear directly in the BF derivation. The assumed adiabatic expansion of the universe is naturally tracked by comoving volumes of constant entropy, and it is this translation to the language of comoving volumes that allows contact with the geometric dSE picture. As we will discuss in Sec. V, adiabaticity is only one of several assumptions required for the simplest version of BF to agree with dSE.

We will show that relaxing the simplifying assumptions behind the BF bound can lead to a physically different scenario with different limits on the length of inflation or predictions for curvature. Only by explicitly requiring a geometric statement of the holographic principle as part of the BF analysis do we ensure that it is actually imposing the same constraint as in the dSE analysis.

In the next section, we will expand the investigation into the assumptions required in order to match the BF and dSE pictures. We will look at more complicated examples consistent with our universe and also more general cases. Considering these cases will reveal some issues that arise when pursuing a rigorous holographic bound valid for all cosmologies.

³ Imposing this requirement at all times is implicit in the BF analysis for simple cosmologies but in general represents an additional assumption.

V. RE-EXAMINING ASSUMPTIONS FOR THE BF AND DSE BOUNDS

A. Overview

Here we give a more detailed account of key assumptions and simplifications that go into the BF and dSE bounds. This will help us examine how these assumptions relate to the equivalence (or not) of the BF and dSE bounds. First we give a descriptive list of these assumptions and then elaborate on each one in separate subsections.

Horizons: Both BF and dSE involve identifying horizons, but there are subtleties in using these horizons that need to be understood in order to make a sharp comparison.

Prefactor: The simple thermodynamic entropy scaling relation of Eqn. (6) has a pre-factor that requires scrutiny.

Net equation of state: The BF picture relies on a fluid with a single equation of state throughout the cosmic evolution. There are choices involved in defining a single effective equation of state for a realistic universe composed of multiple fluids with different equations of state.

Adiabaticity of the fluid: We also need to account for possibly substantial ordinary entropy production that does not necessarily change the cosmological equation of state, such as particle decays or stellar processes.

Black holes: Universes do form black holes, and we need to examine the BF approach of excluding black hole entropy.

TOV equation: We will examine solutions of maximum entropy permissible within a non-collapsing universe with cosmological constant, represented by the Tolman-Oppenheimer-Volkoff equation. This will allow us to generalize beyond the homogeneous.

Alternate cosmologies: We will explore how well the two pictures can describe non-collapsing cosmologies that exhibit a loitering period of slow expansion, allowing observation of an arbitrarily large volume of the universe.

A careful look at each of these issues will help us get a better understanding of the challenges involved in formulating such types of holographic bounds, both for our universe, and in general.

B. Choice of horizons

Banks and Fischler count the entropy at the exit of inflation by modeling the universe as a fluid-filled sphere with a size equivalent to the apparent horizon [3]. In dSE, the past horizon is used, specifically

$$h_P(a_1) \equiv a_1 \int_{a_1}^{a_\Lambda} \frac{da}{a^2 H}. \quad (16)$$

Figure 2 shows the evolution of both these horizons in a way which allows us to visually express the bounds on the number of e-foldings of inflation in both the dSE and BF pictures.

We evaluate the BF bound by requiring that the entropy always be less than the maximal entropy contained within the fluid-filled cavity the size of the apparent horizon. Assuming adiabaticity comoving regions contain constant entropy and can be represented by $R \propto a$ lines in Fig. 5. The heavy dashed line in Fig. 5 (also shown as a dashed line in Fig. 2) indicates the largest comoving region ever contained within the apparent horizon (solid curve), and is thus the natural focus of the BF analysis. To calculate the BF bound on the number of e-foldings geometrically we follow the dashed line in Fig. 2 back to where it intersects the apparent horizon in the inflationary epoch, which marks the earliest allowed start to inflation in the BF picture. We read off the bound on the number of e-foldings as the horizontal distance between this intersection and the end of inflation.

In the dSE picture the universe is bounded by the past horizon which can be well approximated by $h_P \propto a$ (the dotted line in Fig. 4) for most of the evolution of the universe. We can see how this approximation can be used to picture the dSE bound in a similar way to how we just described the BF bound: The earliest allowed start to inflation in the dSE picture is given by where the dotted line in Fig. 4 intersects the apparent horizon in the inflationary epoch.

The $h_P \propto a$ approximation breaks down at both early and late times. In the late time era of cosmological constant domination (shown in detail in Fig. 3), looking back from a later time does not noticeably change the past horizon. This is a nice feature since it means that it does not matter when we choose to observe as long as it is during cosmological constant domination. However the breakdown of the approximation at early times (detailed in Fig. 4) presents some problems for constructing an accurate prescription for a bound on inflation. The past horizon approaches the apparent horizon at early times

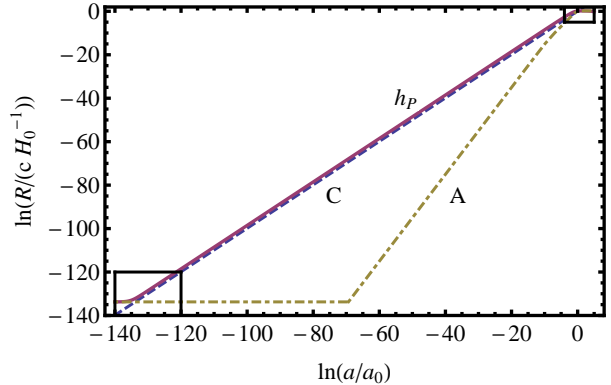


FIG. 2: Evolution of length scales R with scale factor a . Plotted are the size of the apparent horizon A (dot-dashed) and the past horizon h_P of an event late in the de Sitter era (solid). The dashed line C tracks the comoving volume of space with the maximal entropy that can be contained in the fluid cavity of BF assuming adiabaticity. Zoomed versions of the boxed regions are shown in Figs. 3 and 4. (In these zoomed figures the differences between C and h_P appear clearly).

but never crosses it, seemingly implying that there is no bound. However, in the dSE picture we are inclined to say that the deviation from the approximation at early times occurs at a time where we expect new physics and the breakdown of the effective field theory (EFT). Without a clear picture of what lies beyond the EFT we simply use the intersection of dotted line (the $R \propto a$ extrapolation of the past horizon) with the dot-dashed curve (the apparent horizon) to indicate the effective start to inflation.

The distinction between the two horizons seems like a small technical difference, but as we will see in Sec. V H, the distinction can become problematic for universes with large curvature or periods of slow expansion.

C. Thermodynamic relation for entropy density

The expression for the entropy density of an adiabatically expanding fluid, $\sigma = \beta \rho^{\frac{1}{1+w}}$, comes from statistical mechanics. We would like to evaluate the proportionality factor to make this relation more precise. For a particle species in thermodynamic

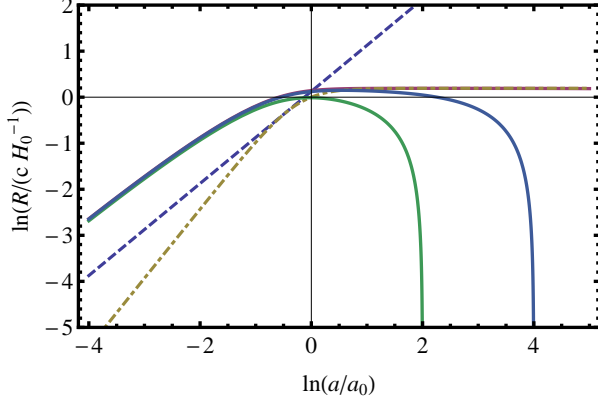


FIG. 3: Close-up of Fig. 2 near the era of cosmological constant domination. The dot-dashed curve is the apparent horizon and the solid curves show past horizons for events at a few different times in the de Sitter era. The feature that these past horizons all approach each other at early times is due their being defined relative to events in the de Sitter era. The dashed line tracks the comoving volume of space with the maximal entropy that can be contained in the fluid cavity of BF assuming adiabaticity.

equilibrium,

$$\rho = \frac{g}{(2\pi)^3} \int E \frac{d^3 p}{e^{(E-\mu)/T} \pm 1} \quad (17)$$

$$P = \frac{g}{(2\pi)^3} \int \frac{p^2}{3E} \frac{d^3 p}{e^{(E-\mu)/T} \pm 1} \quad (18)$$

$$\sigma = \frac{\rho + P}{T}, \quad (19)$$

where ρ is the energy density, P is the pressure, σ is the entropy density, $g = g_B + \frac{7}{8}g_F$ is the total effective number of internal degrees of freedom, and g_B and g_F are the number of spin states for bosons and fermions respectively. For example, in the case of relativistic particles,

$$\rho = g \frac{\pi^2}{30} \frac{(kT)^4}{(\hbar c)^3} \quad (20)$$

$$\sigma = gk \frac{2\pi^2}{45} \left(\frac{kT}{\hbar c} \right)^3 \quad (21)$$

$$\sigma \approx 1.0098 g^{\frac{1}{4}} k (\hbar c)^{-\frac{3}{4}} \rho^{\frac{3}{4}}, \quad (22)$$

and we find that the pre-factor in this case is indeed of order one (as long as the number of internal degrees of freedom is not extraordinarily large). If we have a fluid of relativistic particles at the Planck density in thermal equilibrium, it will be at roughly

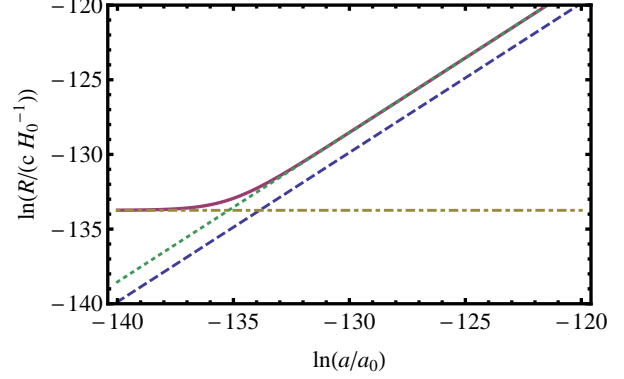


FIG. 4: Close-up of Fig. 2 plotted near the beginning of inflation. Plotted are the size of the apparent horizon (dot-dashed) and the same h_P curves shown in Figs. 2 and 3 (solid). The dashed line tracks the comoving volume of space with the maximal entropy that can be contained in the fluid cavity of BF assuming adiabaticity. The dotted line shows the approximation $h_P \propto a$. The difference between the dotted and dashed lines corresponds to an extra 1.3 e-foldings of inflation more in the dSE picture than in the BF picture.

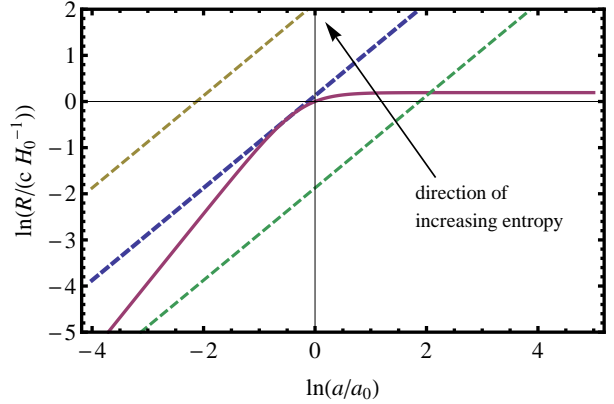


FIG. 5: Evolution of various length scales vs. scale factor a . Plotted are the size of the apparent horizon (solid) and a few lines of constant entropy (dashed). The comoving volume of space with the maximal entropy that can be contained in the fluid cavity of BF assuming adiabaticity is the thick central dashed line tangent to the solid curve.

Planck temperature and if it is contained in a Planck volume, we see from the expression above that it will have of order one unit of entropy as expected.

We will also look at what happens during the early universe in a typical model where a thermal relic decouples non-relativistically. A typical scenario is

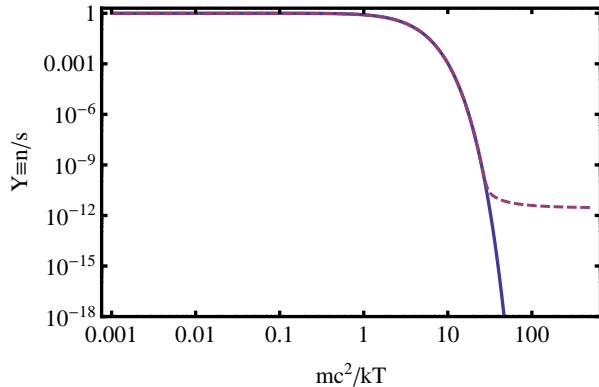


FIG. 6: The generic behavior of entropy density as a function of time during freeze-out. As the universe cools, the number of particles drops until the particles cannot find each other to annihilate and freeze out. The solid line indicates the equilibrium abundance, while the dotted line is the actual particle abundance.

that the inflaton reheats into relativistic particles, some of which are massive and will eventually cool and become non-relativistic. The non-relativistic matter will then typically freeze out and as it does so, asymptote to a constant comoving number density.

In general, non-relativistic matter will possess less entropy than radiation. If we trace the matter’s history back to a time when its temperature was higher than its mass, it had the same entropy per degree of freedom as the relativistic case. However during its transition to a non-relativistic fluid the matter’s comoving number density (and entropy density) exhibits a characteristic drop as it cools before asymptoting to a constant value again during freeze-out (Fig. 6). The entropy from the annihilation of particles is deposited in the radiation. The resulting pre-factor β can be many orders of magnitude smaller than for relativistic matter.

The concept of thermal wavelength is a good way to see the proportionality factors. A fluid in a box the size of the thermal wavelength (not a box of Planck volume) should have entropy of order one. For the same energy density, matter will always have a larger thermal wavelength than radiation.

D. Equation of state

A cosmology like our own universe is not dominated by a single fluid equation of state from reheat-

ing until cosmological constant domination. Even in the simplest one-component case, the transition to an asymptotic de Sitter state modifies the effective equation of state, rounding off the sharp corners in pictures like Fig. 1 and adjusting the crudest estimates for inflation bounds. For example, the simple entropy scaling of Eqn. (6) will begin to fail around the time the dominant fluid energy density drops to near ρ_Λ (see Fig. 3). In addition, the important transition from radiation to matter domination in our universe’s history sits somewhat uncomfortably with the one fluid model of BF.

Perhaps the simplest resolution is to define an effective equation of state for the entire universe’s history (in effect drawing the straight line JL on Fig. 1) and then proceed with the BF formulation. But there is something contrived about this approach. It requires supplying the entire subsequent evolution of the universe as an input to a calculation formally made at the end of reheating. In fact the motivation for this choice is an attempt to keep the BF picture in compliance with a more geometric approach to the holographic principle, along the lines of the dSE picture. If we insisted on calculating the BF bound in our universe using the radiation fluid that dominated after reheating and through most of the expansion history of the universe, we would calculate a larger maximum number of e-foldings of inflation than if we input the actual effective ‘mixed’ equation of state.⁴ In Sec. V E we will see other problematic examples involving changing fluid constituents. It should be pointed out that there is not an immediate “correct” choice for equation of state; no matter how we choose we are forced to decide which clarifications of BF seem most reasonable (or least unappealing). This is a pattern we will see again.

E. Adiabaticity assumptions

Our own universe’s early evolution was well approximated as adiabatic, but even with the exclusion of purely gravitational entropy its subsequent history included substantial non-adiabaticity. And

⁴ Following [16, 17] we note that the change from radiation to matter domination is the reason the CMB entropy is less than the maximal BF bound for radiation. We can estimate the entropy density of the CMB radiation by using the value of ρ_r in Eqn. (22). Since $\rho \propto H_\Lambda^2 = H_0^2 \Omega_\Lambda$ in the BF bound while $\rho_r \propto H_0^2 \Omega_r$, the actual CMB has less entropy than the BF bound by a factor of $\left(\frac{\Omega_\Lambda}{\Omega_r}\right)^{3/4}$.

there is no requirement that similar cosmologies be even as approximately entropy-conserving as ours, as it is no great theoretical challenge to come up with mechanisms to increase entropy. Moreover, the BF bound was derived under the principle of maximizing entropy that could be packed into a sphere without collapsing. One might even wonder whether the maximal entropy non-collapsing solutions indeed are the uniform density solutions assumed so far. (We will further discuss this in Sec. V G and conclude that the homogeneity assumption is OK.) If the universe reheats into a state which does not have maximal entropy (a realistic case), we are forced to decide among multiple interpretations of the BF bound. A treatment of increasing entropy in the Banks and Fischler picture could alter the inflation bound in either direction, depending on how one modifies the BF procedure.

One way to characterize adiabaticity (or its lack) during a cosmology dominated by a single fluid is simply by using the pre-factor β in the expression $\sigma = \beta \rho^{\frac{1}{1+w}}$. In the BF picture for a universe with a fixed effective equation of state but substantial non-adiabaticity, deciding how to handle this pre-factor β is non-trivial.

A simple example of entropy production that is difficult to handle in the BF picture is the production of light from stars, and the subsequent thermalization of that light by dust. This process converts high energy photons into many lower energy ones. The energy density is conserved, yet the temperature decreases. Using $\sigma = \frac{\rho+P}{T}$, we can see that here the entropy will be greatly increased, yet the equation of state remains the same.

For an extreme example of non-adiabaticity, a long period of reheating can have an effective equation of state equivalent to non-relativistic matter, and subsequently upon final exit of reheating gains the stiffer equation of state for radiation. At constant energy densities, $\sigma = \beta \rho^{\frac{1}{1+w}}$ increases enormously. A hypothetical universe dominated by matter with a decay time shorter than t_Λ would be another such scenario. It is not so obvious in either example exactly when one should calculate the post-inflation entropy that is to be compared to the entropy bound for the final state of matter. Whichever choice one makes, any of the non-adiabatic scenarios will have a lowered initial entropy compared to an analogous adiabatic cosmology with similar final matter configuration.

For an illustration of how these choices reflect truly different versions of the BF holographic picture, consider the above example of a universe dom-

inated by unstable matter followed by a decay back to radiation. Inflation would end with

$$S \sim e^{3N_e} H_I^{-3} H_I^{\frac{2}{1+w}} = e^{3N_e} H_I^{-1}, \quad (23)$$

whereas the bound would be⁵

$$S < H_\Lambda^{-3+\frac{2}{1+w}} = H_\Lambda^{-3/2}. \quad (24)$$

This allows

$$3N_e + (-\ln H_I) < (-3/2) \ln H_\Lambda \quad (25)$$

$$N_e < (3/2) \ln \left(\frac{H_\Lambda}{H_I} \right)^{-1} + (1/6) \ln H_I^{-1}, \quad (26)$$

a much larger bound than the one finds in the purely radiation-dominated universe:

$$N_e < (3/2) \ln \left(\frac{H_\Lambda}{H_I} \right)^{-1}. \quad (27)$$

The result of the extra inflation in this calculation is that some radiation produced after reheating never reenters the apparent horizon by the time of Lambda domination, since the subsequent evolution under matter domination does not increase H^{-1} quickly enough to “catch” the biggest scales produced at the start of inflation.

A universe with matter that never reenters the maximal observer’s horizon is one that is physically different from the one described by dSE. One could of course resolve the difference by carefully formulating the BF bound so as to anticipate the exact degree of non-adiabaticity within the cosmological solution chosen. Or one could interpret the larger bound on e-foldings in the BF picture as simply a high estimate, less stringent than could be obtained with more careful analysis. It is worth noting that as formulated the dSE bound is not sensitive to these particular concerns over entropy production. For this reason any attempt to match the BF and dSE bounds more exactly in realistic cosmologies will generally require clarifications or modifications to the setup of BF. There is no guarantee that such modifications will be defensible without the guidance of some organizing principle. We find

⁵ There is entropy production in the decay of the matter, but it is not clear whether it would be enough to saturate the bound for a new equation of state. In particular, this relation assumes that all the matter decayed to radiation, which is then in thermal equilibrium.

the geometric ideas within dSE offer a useful approach.

There is a further problem with accommodating entropy production within the BF framework. We obtain the bound on the e-foldings of inflation using only the entropy at reheating. If we are to accept that entropy only present at a later time can influence the evolution of the universe during inflation, then we see no reason that all sources of future entropy should not similarly affect inflation. These considerations make the BF bound seem rather ad hoc. A geometric picture such as dSE does not suffer from this problem (neither does BF with the requirement of adiabaticity).

F. Black hole formation

In our universe, the dominant contribution to the entropy is the de Sitter horizon, and after that, black holes [18]. If included in the BF calculation, the entropy of black hole formation would completely invalidate the adiabatic approximation, requiring either a very different approach or resulting in a substantially weaker bound on the total number of e-foldings. However, Banks and Fischler explicitly exclude black hole entropy, and the universe is treated as if it were uniform density (effectively replacing the mass in black holes with a contribution to the uniform cosmological energy and entropy density). But if we are to truly ignore black hole entropy as being hidden behind the horizon, then they would instead contribute no entropy at all. Carefully implementing the BF prescription by properly accounting for the hidden black hole matter will therefore *reduce* the counted entropy of our universe with every black hole formed. In any case, in our universe the mass fraction of black holes is small, so this approximation makes no practical difference in the comparison to the dSE case.⁶ But the formal exclusion is a real effect on the entropy counting and thus the calculation of entropy bounds; in comparison, the large-scale geometric approach of the dSE picture is unaffected by local replacement of matter with black holes.

⁶ We leave open the question of potentially substantial differences in universes with a large black hole fraction.

G. Tolman-Oppenheimer-Volkoff equation

We have discussed various methods for estimating the maximum entropy of a *homogeneous* region with a particular fluid equation of state, but Banks and Fischler ask what bounds arise for the most general fluid configuration, and use the Tolman-Oppenheimer-Volkoff (TOV) equation to go beyond the homogeneous case. The TOV equation determines the equilibrium solution which optimizes the amount of entropy which may be stuffed within a volume without collapsing to a black hole.⁷ Here we further extend Banks and Fischler's TOV work to the case of a universe with a cosmological constant.

The TOV equation

$$\frac{dp}{dr} = -\frac{(\rho + p)(Gm(r) + 4\pi Gr^3 p)}{r(r - 2Gm(r))} \quad (28)$$

has families of solutions with different central densities, one of which is the homogeneous solution [23]. Including the cosmological constant, the TOV equation becomes the TOV- Λ equation, where $p = p + p_\Lambda$ and $m(r) = \int_0^r 4\pi r'^2 \rho dr' + 4\pi/3 \rho_\Lambda r^3$.

Our universe is extremely homogeneous on the largest scales, so any non-uniform TOV solution is a poor fit to our universe. However, we would like to know if the BF bound actually favors a different universe. Applying the dSE and BF pictures in these universes may also better illuminate differences between the two approaches.

The solutions to the TOV- Λ equation resemble an Einstein static universe. They are closed static universes of finite size which extend out to the apparent horizon and have the average energy density in matter equal to roughly twice the energy density of the cosmological constant. Figure 7 shows some illustrative solutions.

We find that the inhomogeneity of the TOV solutions does not significantly increase the entropy over the Einstein static universe, so in the end there is no point in considering the inhomogeneous case. Furthermore, the Einstein static universe does not have an asymptotic de Sitter future so we cannot directly apply either the BF or dSE bounds. As we will discuss in the next section, we can however ex-

⁷ Although the TOV equation assumes spherical symmetry and hydrostatic equilibrium, a series of papers [19–22] showed that the TOV solutions are also the maximal entropy solutions for any configuration of a fluid, independent of symmetry (in cases that do not collapse).

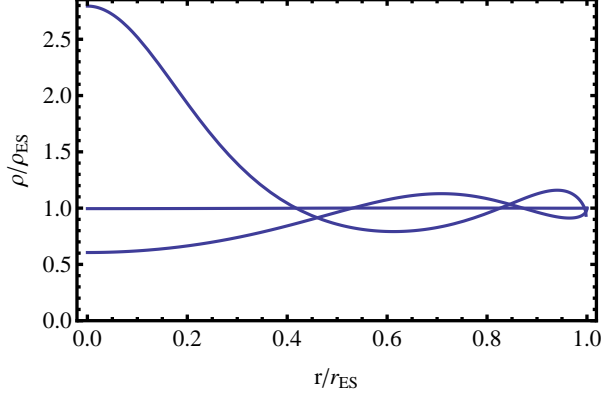


FIG. 7: The variation of energy density as a function of distance coordinate for several TOV- Λ universes. Note how the energy density tends to oscillate around the fixed value that would correspond to the Einstein Static universe. The behavior of these curves is governed by the central density and the equation of state parameter w of the fluid. Fluids with $w \rightarrow 0$ oscillate more than fluids with $w \rightarrow 1$. Shown are curves for a fluid with $w = 0.3$. The straight central line corresponds to the Einstein static universe, which is a member of the family of solutions. These plots are normalized by the energy density of this Einstein static universe (ρ_{ES}) and the apparent horizon of the Einstein static universe (r_{ES}).

amine “loitering universes” with slightly less matter content than the Einstein static universe.

H. Other cosmologies

In other cosmologies, the discrepancy between the dSE and BF bounds can become more pronounced. In the case of a loitering universe (Fig. 8) the Hubble constant becomes very small for an extended period of time before heading on to de Sitter expansion. During the loitering phase of slow expansion, these are approximately Einstein Static universes and their large curvature causes the Hubble length to deviate substantially from the apparent horizon. The slow expansion causes the Hubble horizon and consequently the past horizon to become very large, as shown in Fig. 9; however, the apparent horizon is largely unaffected as shown in Fig. 10.

If we fine tune the ratios of densities of fluid components and the cosmological constant, an observer can see an arbitrarily large volume of the universe within her past horizon. This is troublesome for holographic ideas in general, and especially we see that the dSE picture would have an arbitrarily weak

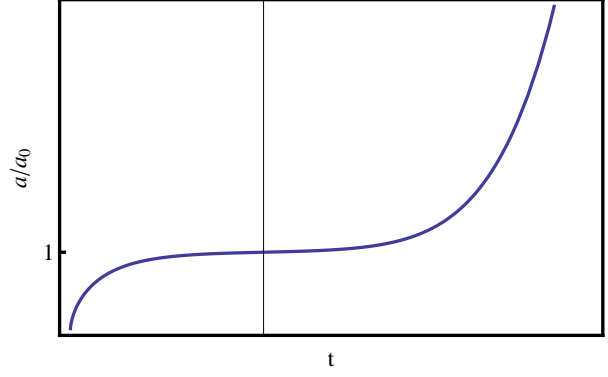


FIG. 8: The evolution of the scale factor vs. time for a loitering universe. Because the expansion slows down during the loitering phase, the Hubble length becomes very large. Here a_0 is set arbitrarily at the inflection point.

limit on e-foldings in an arbitrarily fine-tuned loitering cosmology. Furthermore, the picture of BF would still have a strong bound on e-foldings, in spite of being able to observe an arbitrarily large region of space. This would indicate an arbitrarily large observable entropy⁸, in spite of reaching an asymptotic de Sitter final state. In our view neither the BF nor the dSE pictures as currently described in the literature can be applied without further elaboration to the loitering cosmologies. These cosmologies, while they do not describe our universe, are interesting and useful as test cases because they allow much more entropy to be observed while still possessing the same final asymptotic de Sitter state. And it is important to note that even a small-curvature universe of the correct sign has a little bit of this loitering behavior and the more finely tuned solutions are smoothly connected to ours in model parameters. Thus, in a systematic application of these holographic principles to some ensemble of cosmologies the dSE and possibly also BF approaches could favor the loitering direction.

⁸ There would be negligible cosmological redshift within an arbitrarily large observable region because of the slow expansion during the loitering phase.

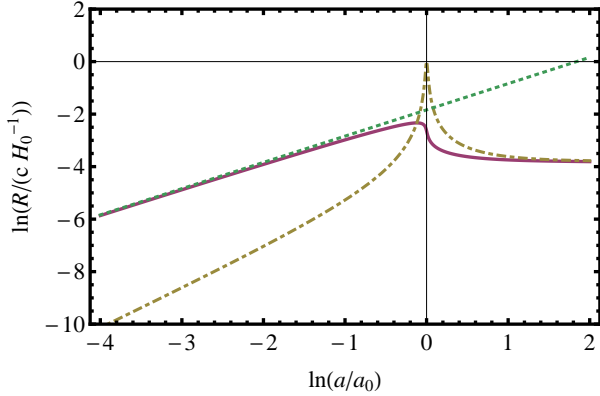


FIG. 9: Evolution of length scales R for the loitering universe shown in Fig. 8 vs. a shown near the era of cosmological constant domination. Plotted are the Hubble length (dot-dashed) and the past horizon h_P of an event late in the de Sitter era (solid). The dotted line shows the approximation $h_P \propto a$. Notice that the loitering phase causes the horizons to be larger, allowing more of the universe to be observed. The upturns in the curves of Hubble length and past horizon near $a = a_0$ depend on the duration of the loitering phase, and can be increased arbitrarily (though the level of fine tuning also increases similarly).

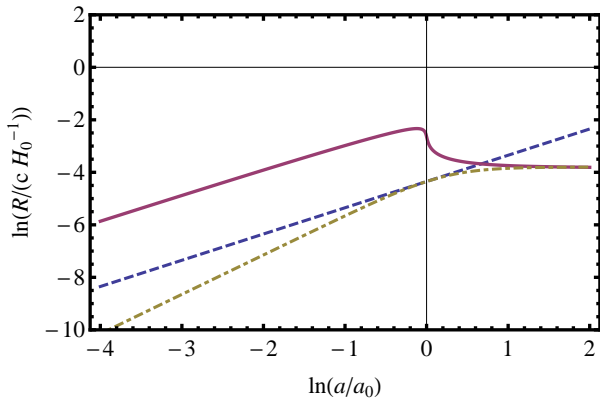


FIG. 10: More information about the same loitering universe shown in Fig. 9, also plotted vs. scale factor a and shown near the era of cosmological constant domination. Plotted are the apparent horizon (dot-dashed) and the past horizon of an event late in the de Sitter era (solid). The dashed line tracks the comoving volume of space with the maximal entropy that can be contained in the fluid cavity of BF assuming adiabaticity. Increasing the duration of the loitering phase of this cosmology increases the allowed e-foldings in dSE arbitrarily, while having little effect on the BF bound. This illustrates how the BF and dSE bounds can become arbitrarily different in a loitering universe.

VI. SHOULD THE BOUNDS BE SATURATED?

We have directed our analysis to the task of comparing upper bounds on inflation that arise in the BF and dSE pictures for cosmology. A separate question is: how close to this bound should we expect a universe described by one of these pictures to be? In the dSE picture there are some reasons to expect a typical universe to be near the bound; essentially, the mechanism starting inflation makes it *more* likely to start high on the inflaton potential. In the BF picture things are less clear. Its statement as a bound on entropy encourages us to think about maximization because of our experience with the second law of thermodynamics. But we are also used to thinking of the universe as having an extremely low entropy initial state. It is actually only because BF exclude gravitational and horizon entropy that we can even begin thinking about the universe as being near a kind of entropy maximum. Without additional principles (such as we have in the dSE case) we do not see a particular reason that the BF bound need be near saturation.

A closely related question is: if we interpret observations of our own universe within the BF or dSE picture, should we expect the inflation experienced by our own universe to saturate these bounds? The added wrinkle is that observations have already established an effective floor on the number of e-foldings. Thus the typical bound on inflation within the BF picture or the dSE picture amounts to “just a few e-foldings more than what we have observed”. This apparent coincidence is really the coincidence that we are living at a time that makes us nearly “maximal observers” so we can already see most of what we will ever see. At least in the BF case, it is observation-based priors that drive us close to saturating the bound. More generally both of these models have the feature that the bounds are not far above the minimum amount of inflation we expect based on observations.

VII. CONCLUSIONS

We have shown that the entropy bound of Banks and Fischler and that of dSE coincide for a very restrictive set of assumptions and a simplified cosmology. Yet closer investigation reveals that even this result requires approximations within the models, and indeed the conceptual and practical differences between them are minimized by the choice of cos-

mology. Attempting to perform the comparison on a cosmology more closely resembling our own (with its multiple equations of state or failures of adiabaticity) raises many technical issues that in aggregate call into question how fundamental the correspondence is between these two approaches.

Examining even more exotic cosmologies as test cases merely heightens these issues, and moreover shows that the project of implementing either approach as a consistent, rigorous principle across cosmologies is not quite as straightforward as it might appear. While the phrasing of the BF bound in terms of entropy sounds pleasingly universal, the details of its implementation rely heavily on the cosmological history of the universe to which it applies. As we have seen while attempting this implementation, it is roughly possible to map the BF picture onto the dSE picture by carefully working backwards to en-

tropy from geometric notions in which the dSE picture is originally phrased. Because these geometric ideas are more robust under variations in cosmological history, we ultimately find them a more practical and compelling basis for formulating a predictive holographic principle for finite universes with inflation. Moreover, the unexpected complexities arising from examining unusual cosmologies such as the loitering universes suggest a need to further sharpen such a principle.

Acknowledgments

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